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Photonic Pseudogaps for Periodic Dielectric Structures

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We consider the problems of existence and structure of gaps (pseudogaps) in the spectra associated with Maxwell equations and equations that govern the propagation of acoustic waves in periodic two-component media. The dielectric constant ε is assumed to be real and positive, and the value of $\varepsilon=\varepsilon_b$ on the background is supposed to be essentially larger than the value of $\varepsilon=\varepsilon_a$ on the embedded component. We prove the existence of pseudogaps in the spectra of the relevant operators. In particular, we give an accurate treatment of the term "pseudogap." We also show that if the contrast $\varepsilon_b/\varepsilon_a$ approaches infinity, then the bands of the spectrum shrink to a discrete set which can be identified with the set of eigenvalues of a Neumann-type boundary value problem and thus can be effectively calculated.

KEY WORDS: Waves; periodic dielectrics; periodic acoustic media; pseudogaps in the spectrum.

INTRODUCTION

The idea of finding and designing periodic and disordered dielectric materials which exhibit respectively gaps in the spectrum or localized modes was introduced quite recently.[1-3] The hope to find such disordered media for electromagnetic waves is based on the remarkable Anderson localization phenomenon[4] for the propagation of electron waves in a disordered solid. The general reason for the rise of gaps or localization lies in the coherent multiple scattering and interference of waves (see, for instance, John[5] and references therein). The experimental results[6-8] for periodic and disordered dielectrics indicate that the photonic gap regime and corre-

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spondingly light localization can be achieved for some nonhomogeneous materials. The analysis of some approximate models and the numerical computations[9-14] have shown the possibility of a gap (or pseudogap) regime for some two-component periodic dielectrics. The most recent theoretical and experimental achievements in the investigation of the photonic band-gap structures are published in the series of papers in ref. 15.

To study the properties of wave propagation in a nonhomogeneous medium one has to investigate the spectral properties of the relevant self-adjoint differential operators with coefficients varying in space. Such an operator for electromagnetic waves has the form

$$\Lambda\Psi = \nabla \times (\varepsilon^{-1}(x)\, \nabla \times \Psi), \qquad x \in \mathbf{R}^3$$

where $\Psi(x)$ is a complex vector function on $\mathbf{R}^3$. An important analog of this operator is the following operator of second order acting on the space of complex scalar-valued functions $\psi(x)$ on $\mathbf{R}^d$:

$$\Gamma\psi = -\sum_{j=1}^{d} \frac{\partial}{\partial x_j} \varepsilon^{-1}(x) \frac{\partial}{\partial x_j} \psi, \qquad x \in \mathbf{R}^d$$

This operator can be associated with the propagation of acoustic waves. In these formulas $\varepsilon(x)$ stands for the electric permittivity for electromagnetic waves, whereas for acoustic waves it stands for the mass density of the medium. The coefficient $\varepsilon(x)$, $x \in \mathbf{R}^3$, we consider here is a periodic field bounded from above and below by positive constants. If $\varepsilon(x)$ is a random field which is a small perturbation of a positive periodic field $\varepsilon_0(x)$, i.e., $\varepsilon(x) = \varepsilon_0(x) + \varepsilon_1(x)$, where $\varepsilon_1(x)$ is a small random field, then according to the philosophy of the Anderson localization, one may expect the rise of localized states for the random field $\varepsilon(x)$ in the gaps of the spectrum of the relevant operator associated with the periodic $\varepsilon_0(x)$. This justifies the special interest in periodic structures and their simplest realization in two-component periodic media. Namely, the important parameters of such a two-component periodic medium which can shape the spectrum[8] are the volume-filling fraction, the dielectric constant contrast $\varepsilon_b/\varepsilon_a$ (where ε_b and ε_a are, respectively, the dielectric constants of the host material and the embedded components), and the shape of atoms of the embedded material as well as their arrangement. In particular, the high dielectric constant contrast favors the rise of gaps in the spectrum (some living tissues possess very high contrast[16]).

We give a definition of the term "pseudogap" and prove the existence of pseudogaps for two-component dielectrics (or acoustic media) which can be thought of as bubbles of air embedded in an optically dense background. Thus, we consider the media with high contrast in the dielectric constant. Under the assumption that the contrast approaches infinity, we also find the precise limit location of the bands of the spectrum, which turns out to be a set of eigenvalues of the Neumann-type boundary value problem associated with a bubble of air. This gives some rough estimates for where the gaps of the spectrum can be and confirms the dependence of the structure of the spectrum on the shape of the bubbles. The rigorous proof of the existence of true gaps in the spectrum for the finite-difference versions of the operators Λ and Γ was obtained in ref. 17. The sketch of the proof of the existence of true gaps for the operators Λ and Γ under some extra conditions is given in ref. 22.

1. STATEMENT OF RESULTS

To study the properties of wave propagation in a nonhomogeneous medium it is important to investigate the spectral properties of the relevant self-adjoint differential operators with coefficients varying in space. The operators of interest are

$$\Lambda\Psi = \nabla\times(\gamma(x)\,\nabla\times\Psi), \qquad \gamma(x)=\varepsilon^{-1}(x), \qquad x\in\mathbf{R}^3 \tag{1}$$

$$\Gamma\psi = -\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\gamma(x)\frac{\partial}{\partial x_j}\psi, \qquad x\in\mathbf{R}^d \tag{2}$$

In these formulas $\Psi(x)$ is an $\mathbf{R}^3(\mathbf{C}^3)$-valued vector function, $\psi(x)$ is a real- (complex-) valued scalar function, and the operator Λ is the operator associated with the propagation of electromagnetic waves, whereas the operator Γ can be viewed as its analog for the case of scalar-valued wave functions. In particular, if $d=3$, Γ can be associated with the propagation of acoustic waves. We begin with a construction of such a two-component medium in the space $\mathbf{R}^d$ for which the coefficient $\varepsilon(x)$, $x\in\mathbf{R}^d$, takes on value 1 on a set of disjoint bounded finite domains (a sort of air bubbles) spread in the space, and it takes on a value grater than 1 in the rest of the space, which we call the background, so $\varepsilon(x)\geqslant 1$ for all x. We shall call these domains atoms. In fact, we will be interested in the case when ε tends to infinity on the background, that is, in the medium with a high contrast in ε on the background and the atoms. In order to describe the medium accurately suppose that the space $\mathbf{R}^d$ contains a set of open bounded domains O_α (which do not overlap) with boundaries ∂O_α, respectively, where index α runs over a set of indices Z (it could be the set of natural numbers or the lattice $\mathbf{Z}^d$ for periodic structures). We pick the standard orientation for each ∂O_α, that is, the normal vector ν points toward the exterior of O_α. Thus, if we denote the union of O_α by $\mathscr{A}$ and its complimentary set that forms the background by $\mathscr{B}$, then we have

$$\bigcup_{\alpha \in Z} O_\alpha = \mathscr{A}, \qquad O_\alpha \bigcap O_\beta = \varnothing \quad \text{if} \quad \alpha \neq \beta; \qquad \mathscr{B} = \mathbf{R}^d - \mathscr{A}$$

In order to consider the limit case when ε tends to infinity on the background $\mathscr{B}$, we introduce ε which depends on a parameter $s > 1$ in the following way:

$$\varepsilon = \varepsilon(s, x) = \begin{cases} 1 & \text{if} \quad x \in \mathscr{A} \\ s & \text{if} \quad x \in \mathscr{B} \end{cases} \tag{3}$$

We shall assume the boundaries of the domains O_α to be regular in the following sense.

Definition. Let Ω be an open domain in $\mathbf{R}^d$. We shall say that the boundary $\partial\Omega$ is regular if it is either smooth (from the class C^∞) or if it is a parallelepiped.

In fact, one may consider more general conditions on the smoothness of the boundary, but for the sake of simplicity we shall deal with regular domains by means of the definition above.

Now let us suppose that O is one of the domains O_α with a boundary ∂O and consider the following self-adjoint operators:

$$\Lambda_O \Phi = \nabla \times (\nabla \times \Psi), \qquad x \in O, \qquad \nu \times (\nabla \times \Psi)|_{\partial O} = 0 \tag{4}$$

$$\Gamma_O \psi = -\nabla \psi, \qquad x \in O, \qquad \frac{\partial}{\partial \nu} \psi|_{\partial O} = 0 \tag{5}$$

which act on the Hilbert spaces $L_3^2(O)$ and $L^2(O)$, respectively (the subindex 3 in the first space stands for the dimension of the point values of the weave function Ψ). Thus, the operator Γ_O can be identified with the classical Neumann boundary value problem for the domain O, whereas the respective boundary value problem associated with the operator Λ_O can be viewed as its analog for the differential operation $\nabla \times (\nabla \times (\cdot))$.

The following statement concerning a periodic dielectric medium (for instance, bubbles of air distributed periodically in an optically dense background) holds:

Theorem 1. Suppose that:

(i) $O_a = O + \alpha$, $\alpha \in \mathbf{Z}^d$, where O is an open bounded domain with a regular boundary ∂O, and there exists a positive constant d such that $\text{dist}(O_\alpha, O_\beta) \geqslant d$, if $\alpha \neq \beta$.

(ii) The function $\varepsilon(s, x)$ is defined by (3).

Let $\sigma(\Lambda_O)$ and $\sigma(\Gamma_O)$ be the spectra, respectively, of the operators Λ_O and Γ_O (these spectra are discrete sets), and $E(A, d\lambda)$ stands for the resolution of identity of the self-adjoint operator A. If J is an arbitrary interval containing no points of the spectrum $\sigma(\Lambda_O)$ or $\sigma(\Gamma_O)$, then the following limit equalities hold:

$$\lim_{s \to \infty} E(\Lambda, J)\, \Psi = 0, \qquad \lim_{s \to \infty} E(\Gamma, J)\, \psi = 0 \tag{6}$$

where Ψ and ψ are arbitrary vectors from the corresponding Hilbert spaces.

The relationship (6) can be interpreted as the existence of pseudogaps in the spectrum of the operators Λ and Γ if ε is large on the background.

If O is a cube, the spectra $\sigma(\Lambda_O)$ and $\sigma(\Gamma_O)$ can be effectively found.

Proposition 2. Let O be a cube in d-dimensional space with the edge of length L and $\sigma_{d,L}$ be the spectrum of the relevant Neumann problem (6). Then

$$\sigma_{d,L} = \{\pi^2 L^{-2} k^2, k \in \mathbf{Z}^d\} \tag{7}$$

and

$$\sigma(\Lambda_O) = \sigma_{3,L}, \qquad \sigma(\Gamma_O) = \sigma_{d,L} \tag{8}$$

The equality (7) is a well-known classical fact, and (8) will be derived later.

In the case when O_α are periodically distributed identical cubes the statement of Theorem 1 continue to hold even if the filling fraction of the cubes approaches 1.

Theorem 3. Suppose that the following conditions are satisfied:

(i) O_α, $\alpha \in \mathbf{Z}^d$, are identical cubes with edge of the length L and $O_\alpha = O + (L + l)\,\alpha$, $\alpha \in \mathbf{Z}^d$, where $l = l(s) > 0$.

(ii) The function $\varepsilon(s, x)$ is defined by (3).

(iii) The following relationships are true:

$$\lim_{s \to \infty} l(s) = 0, \qquad \lim_{s \to \infty} s l(s) = \infty \tag{9}$$

Then for arbitrary interval J containing no points of the spectrum $\sigma_{3,L}$ or $\sigma_{d,L}$ we have, respectively, the relationships (6).

Theorems 1 and 3 are consequences of the resolvent convergence of operators Λ_s and Γ_s. Namely, let us introduce the following operators:

$$\Lambda^{(0)}=\left(\otimes \sum_{\alpha \in Z} \Lambda_{O_\alpha}\right)\oplus \Lambda_{\mathscr{B}}, \qquad \text{where} \quad \Lambda_{\mathscr{B}}\Psi \equiv 0, \quad \Psi \in L_3^2(\mathscr{B}) \tag{10}$$

$$\Gamma^{(0)}=\left(\otimes \sum_{\alpha \in Z} \Gamma_{O_\alpha}\right)\oplus \Gamma_{\mathscr{B}}, \qquad \text{where} \quad \Gamma_{\mathscr{B}}\Psi \equiv 0, \quad \Psi \in L^2(\mathscr{B}) \tag{11}$$

which act on $L_3^2(\mathbf{R}^3)$ and $L^2(\mathscr{R}_1^d)$ (the symbol $\oplus$ stands for the direct sum of operators).

Theorem 4. Suppose that the conditions of Theorem 1 are satisfied. If Λ_s and Γ_s stand for the operators Λ and Γ associated with $\varepsilon(s, x)$, respectively, then the following limits (in the strong resolvent sense) hold:

$$\lim_{s \to \infty} \Lambda_s = \Lambda^{(0)} \tag{12}$$

$$\lim_{s \to \infty} \Gamma_s = \Gamma^{(0)} \tag{13}$$

These equalities hold as well if the conditions of Theorem 3 are fulfilled. In this case we drop the operators $\Lambda_{\mathscr{B}}$ and $\Gamma_{\mathscr{B}}$ from the representations for the operators $\Lambda^{(0)}$ and $\Gamma^{(0)}$, respectively.

2. PROOF OF THE RESULTS

We begin with some informal arguments which indicate the validity of Theorems 1 and 3 for the operator Γ. It is well known that the operator Γ can be associated also with the conductance of heat where the function $\gamma(x)$ is the position-dependent heat conductivity. If the value s of $\varepsilon(x)$ on the background gets large, then the effective conductance $\tilde{\gamma}$ of a slab of the background material of thickness l is of the order $\tilde{\gamma} \sim (sl)^{-1}$. By the conditions of the both Theorems 1 and 3 this effective conductance must tend to infinity. Based on this we may expect the following: (i) for large s the heat does not propagate between the atoms of the embedded material; (ii) heat is reflected by the boundaries of these atoms, i.e., the Neumann boundary condition holds. These statements correspond exactly to the statements of Theorems 1 and 3 for the operator Γ. The rigorous arguments we proposed below employ those observations and they are applicable for both Γ and Λ operators.

We notice first that since the function $\gamma(x)$ is discontinuous, then the operators Λ and Γ defined, respectively, by (1) and (2) are self-adjoint by means of corresponding bilinear forms associated with the relevant differential expressions. For the second-order differential operators (in particular, the operator Γ) this is established, for instance, in ref. 18. As far as the operator Λ is concerned, we notice that all arguments used in the mentioned monograph are evidently applicable to the operator Λ with some minor modifications. In particular, the bilinear form associated with the operator Λ is $\int \gamma\, |\nabla \times \Phi(x)|^2\, dx$. If we wish to find the action of those operators on smooth functions Φ and φ, respectively, i.e., in the classical sense, we ought to consider the ones satisfying appropriate conditions of the continuity of the wave functions φ and Φ and their derivatives on the surfaces of the discontinuity $\mathscr{D}_\gamma$ of $\gamma(x)$. In particular, φ and Φ must be continuous, and if ν is the normal vector on $\mathscr{D}_\gamma$, then $\nu \cdot \gamma\nabla\varphi$ must take equal values on both sides of $\mathscr{D}_\gamma$ and the same is true for $\nu \times (\gamma\nabla \times \Phi)$.

The proofs of the theorems for the operators Λ and Γ are analogous and they will be considered simultaneously. Since the operator Γ is an elliptic one and somewhat easier to deal with, we shall consider it first. In many cases the arguments used for the operator Γ hold for the operator Λ with some minor changes, and if not, we provide appropriate arguments for the operator Λ specifically. We shall prove first the basic Theorem 4. In order to do this, we need to prove some auxiliary lemmas.

Lemma 1. Let A_n, $n \in \mathbf{N}$ ($\mathbf{N}$ is the set of natural numbers) and A be self-adjoint nonnegative operators in a Hilbert space H with domains $D(A_n)$ and $D(A)$, respectively. For a densely defined linear operator B let us denote by $\bar{B}$ its closure. We assume the following conditions to be satisfied:

(i) There exists a linear subspace D which is dense in H and is a core of the operator A, i.e., $\overline{A|_D} = A$.

(ii) For any $\psi \in D$ there exists a sequence ψ_n, $n \in \mathbf{N}$, such that $\psi_n \in D(A_n)$ and

$$\lim_{n \to \infty} \|\psi - \psi_n\| = 0, \qquad \lim_{n \to \infty} \|A\psi - A_n \psi_n\| = 0 \tag{14}$$

Then A_n converges to A in the strong resolvent sense.

Proof. We notice first that (14) implies the existence of the strong graph limit of A_n, i.e., st.gr.-lim $A_n = \hat{A}$ is a closed symmetric operator.[19] The relationships (14) also imply evidently $A\psi = \hat{A}\psi$, $\psi \in D$, and, therefore, in view of the condition (i), $\hat{A}$ is an extension of $\bar{A} = A$. Since A is a self-adjoint operator and $\hat{A}$ is symmetric, we may conclude that $A = \hat{A}$. Thus, st.gr.-lim $A_n = A$. This fact together with the self-adjointness of all operators

A_n, $n \in \mathbf{N}$, and A implies that the A_n converge to A in the strong resolvent sense.[19] $\square$

We adopt here the following notations:

For a measurable set $\Omega \in \mathbf{R}^d$, $|\Omega|$ is its Lebesgue measure.

For a bounded m-dimensional surface Γ in $\mathbf{R}^d$, $|\Gamma|_m$ is its area.

For a domain Ω, $\bar{\Omega}$ is its closure.

For a domain Ω, $\partial\Omega$ is its boundary, $v = v(x)$, $x \in \partial\Omega$, is the normal unit vector to $\partial\Omega$.

∂_j, ∂_v, and ∂^α [$\alpha = (\alpha_1, ..., \alpha_d)$, where α_j are nonnegative integers] are, respectively, the partial derivatives $\partial/\partial x_j$, $\partial/\partial v$, and $\partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$.

$C^m(\mathbf{R}^d)$ is the set differentiable functions up to the mth order.

$C_0^\infty(\mathbf{R}^d)$ is the set of infinitely differentiable functions with bounded support.

$\|\varphi\|_{p,\Omega} = (\int_\Omega |\varphi(x)|^p\, dx)^{1/p}$; $\|\varphi\|_p = \|\varphi\|_{p,\mathbf{R}^d}$.

$L^2(\Omega)$ is the Hilbert space of scalar functions on Ω with finite $\|\cdot\|_{2,\Omega}$-norm.

$L_3^2(\Omega)$ is the Hilbert space of $\mathbf{R}^3(\mathbf{C}^3)$-vector functions on Ω with finite $\|\cdot\|_{2,\Omega}$-norm.

To regularize functions we introduce in a standard fashion a mollifier $k: \mathbf{R} \mapsto \mathbf{R}$ satisfying the following conditions:

(i) $k(x) = k(|x|) \geqslant 0$, $k \in C^\infty(\mathbf{R})$

(ii) $k(x) = 0$, $|x| \geqslant 1/2$, $\int k\, dx = 1$.

Then we define a mollifier $K(x)$: $\mathbf{R}^d \mapsto \mathbf{R}$ by the formula

$$K(x) = \prod_{1 \leqslant j \leqslant d} k(x_j)$$

Now for any real-valued function $f(x)$, $x \in \mathbf{R}^d$, and a positive number δ we define the following as its transformation:

$$f_\delta(x) = \delta^{-d} f(x/\delta)$$

In particular, we shall consider the mollifier $K_\delta(x)$ associated with the function K. We also shall use for the standard regularization the convolution $K * f$ of two functions K and f on $\mathbf{R}^d$, namely

$$K * f = \int_{\mathbf{R}^d} K(x - y)\, f(y)\, dy$$

Besides, for any positive a and any measurable set $\Omega \subseteq \mathbf{R}^d$ we define

$$\Omega_a = \{x \in \mathbf{R}^d : \operatorname{dist}(x, \Omega) < a/2\}$$

Lemma 2. Let Ω be an open bounded domain. Then for any $\delta > 0$ there exist a real-valued function $\psi_\delta \in C_0^\infty(\mathbf{R}^d)$ such that:

(i) $\psi_\delta(x) \geqslant 0$, $x \in \mathbf{R}^d$; $\psi_\delta(x) = 1$, $x \in \Omega$; $\psi_\delta(x) = 0$, $x \in \mathbf{R}^d - \Omega_\delta$.

(ii) For any natural q there exists a positive constant $C = C(q, \Omega)$ for which the following inequalities are true for any multiindex α:

$$\|\partial^\alpha \psi_\delta\|_2 \leqslant C\delta^{-|\alpha|} \, |(\partial\Omega)_{2\delta}|, \qquad 1 \leqslant |\alpha| \leqslant q \tag{15}$$

In particular, if $\partial\Omega$ is a piecewise smooth surface, the above inequality can be replaced by

$$\|\partial^\alpha \psi_\delta\|_2 \leqslant C_s \delta^{1-|\alpha|} \, |\partial\Omega|_{d-1}, \qquad 1 \leqslant |\alpha| \leqslant q \tag{16}$$

Proof. Let χ_{Ω_δ} be the characteristic function of the domain Ω_δ, i.e.,

$$\chi_{\Omega_\delta}(x) = \begin{cases} 1 & \text{if } \ x \in \Omega_\delta \\ 0 & \text{otherwise} \end{cases}$$

Then we introduce functions $\psi_\delta = K_\delta * \chi_{\Omega_\delta}$, $\delta > 0$, and notice that the following relationships are true:

$$\begin{aligned} \partial^\alpha \psi_\delta = (\partial^\alpha K_\delta) * \chi_{\Omega_\delta} &= \delta^{-|\alpha|} (\partial^\alpha K)_\delta * \chi_{\Omega_\delta} \\ \psi_\delta(x) = 1 \qquad & \text{if } \ x \in \Omega \\ \psi_\delta(x) = 0 \qquad & \text{if } \ x \in \mathbf{R}^d - \Omega_{2\delta} \\ \partial^\alpha \psi_\delta(x) = 0 \qquad & \text{if } \ x \notin (\partial\Omega)_{2\delta} \end{aligned}$$

This implies

$$\|\partial^\alpha \psi_\delta\|_2 = \delta^{-|\alpha|} \, \|(\partial^\alpha K)_\delta * \chi_{\Omega_\delta}\|_2 \leqslant \delta^{-|\alpha|} \, \|\partial^\alpha K\|_1 \, |(\partial\Omega)_{2\delta}|$$

The last inequality immediately implies (15). The inequality (15) in turn evidently implies (16). This completes the proof of the lemma. ∎

The statement below is a straightforward consequence of Lemma 2.

Corollary 3. Let Ω be a bounded domain with a piecewise smooth boundary and $\varphi \in C^n(\mathbf{R}^d)$; then for any $\delta > 0$ and for a nonnegative integer q, $q \leqslant n$, there exist a positive constant $C = C(q, \Omega)$ such that the following inequalities are true:

$$\|\partial^\alpha(\varphi - \psi_\delta \varphi)\|_2 \leqslant C \, |\partial\Omega|_{d-1} \sum_{\beta \leqslant \alpha} \delta^{1-|\beta|} \, \|\partial^{\alpha-\beta}\varphi\|_\infty, \qquad 0 \leqslant |\alpha| \leqslant q \tag{17}$$

Lemma 4. Let Ω be an open bounded domain with a regular boundary and real-valued functions $\gamma(x)$, $\varphi(x)$, $x \in \mathbf{R}^d$, satisfy the following conditions:

(i) There are constants γ_i, $\gamma_e: \gamma(x) = \gamma_i$, if $x \in \Omega$; $\gamma(x) = \gamma_e$, if $x \in \mathbf{R}^d - \Omega$.

(ii) $\varphi \in C^2(\mathbf{R}^d)$, $\partial_\nu \varphi|_{\partial\Omega} = 0$.

Then the following identity holds:

$$\sum_{j=1}^{d} \partial_j(\gamma \partial_j \varphi) = \gamma \sum_{j=1}^{d} \partial_j^2 \varphi \tag{18}$$

where the derivative on the left side is understood as the weak derivative (see ref. 18).

Proof. Clearly $\gamma(x)$ is discontinuous on $\partial\Omega$, so the derivative on the left of (18) generally speaking could contain relevant Dirac delta-functions. But because of the condition (ii) this does not occur and this derivative is a regular function which equals the function on the right side of (18). To prove this we must show that

$$\int_{\mathbf{R}^d} \sum_{j=1}^{d} (\partial_j \psi)(\gamma \partial_j \varphi)\, dx = \int_{\mathbf{R}^d} \psi \gamma \sum_{j=1}^{d} \partial_j^2 \varphi\, dx \tag{19}$$

for any $\psi \in C_0^\infty(\mathbf{R}^d)$. Let us consider the left integral in (19) and represent it as the sum of two integrals over the sets Ω and $\mathbf{R}^d - \Omega$ where function γ equals the constant γ_i and γ_e correspondingly. Then employing the divergence theorem in a standard manner and the condition (ii) we easily come up with (19). Thus the lemma is proved. ∎

The analogous statement holds for a vector function $\Phi(x)$, $x \in \mathbf{R}^3$. Namely, the following lemma is true.

Lemma 5. Let Ω be an open bounded domain with a regular boundary and real-valued functions $\gamma(x)$, $\Phi(x)$, $x \in \mathbf{R}^d$, satisfy the following conditions:

(i) There are constants γ_i, $\gamma_e: \gamma(x) = \gamma_i$, if $x \in \Omega$; $\gamma(x) = \gamma_e$, if $x \in \mathbf{R}^d - \Omega$.

(ii) $\Phi \in C^2(\mathbf{R}^d)$, $\nu \times (\nabla \times \Phi)|_{\partial\Omega} = 0$. Then the following identity is true:

$$\nabla \times (\gamma(\nabla \times \Phi)) = \gamma(\nabla \times (\nabla \times \Phi))$$

where the derivative on the left side is understood as the weak derivative.

Proof. The proof is analogous to that of the lemma. Namely we show that

$$\int_{\mathbf{R}^d} (\nabla \times \Psi) \cdot \gamma(\nabla \times \Phi)\, dx = \int_{\mathbf{R}^d} \gamma \Psi \cdot (\nabla \times (\nabla \times \Phi))\, dx$$

for any $\Psi \in C_0^\infty(\mathbf{R}^d)$. We use the same argument as in the previous lemma. The only difference is that here we employ the identity $\Phi \cdot (\nabla \times \Psi) - \Psi \cdot (\nabla \times \Phi) = \nabla \cdot (\Psi \times \Phi)$ and the condition (ii) of this lemma. ■

Lemma 6. Let Ω, $\gamma(x)$, $\varphi(x)$, and $\Phi(x)$ satisfy the conditions of Lemmas 4 and 5 and Λ and Γ be the corresponding operators defined by (1) and (2) by means discussed at the beginning of the section. Then there exist functions φ_δ, $\Phi_\delta \in C^\infty(\mathbf{R}^d)$, $\delta > 0$, and a constant C such that

$$\varphi_\delta(x) = 0 \quad \text{if} \quad x \notin \Omega_\delta, \qquad \|\chi_\Omega \Gamma\gamma - \Gamma\varphi_\delta\|_2 \leqslant C\gamma_e \delta^{-1}, \qquad \delta > 0$$

$$\Phi_\delta(x) = 0 \quad \text{if} \quad x \notin \Omega_\delta, \qquad \|\chi_\Omega \Lambda\Phi - \Lambda\Phi_\delta\|_2 \leqslant C\gamma_e \delta^{-1}, \qquad \delta > 0$$

In addition,

$$\lim_{\delta \to 0} \|\varphi - \varphi_\delta\|_2 = 0, \qquad \lim_{\delta \to 0} \|\Phi - \Phi_\delta\|_2 = 0 \tag{20}$$

Proof. Let us define $\varphi_\delta = \psi_\delta \varphi$ and $\Phi_\delta = \psi_\delta \Phi$, where the functions ψ_δ are defined in Lemma 2. Notice first that

$$\varphi(x) = \varphi_\delta(x), \qquad x \in \Omega: \quad \partial_\nu \varphi = \partial_\nu \varphi_\delta = 0|_{\partial\Omega} \tag{21}$$

$$\Phi(x) = \Phi_\delta(x), \qquad x \in \Omega; \qquad \nu \times (\nabla \times \Phi) = \nu \times (\nabla \times \Phi_\delta) = 0|_{\partial\Omega} \tag{22}$$

From this and Lemmas 4 and 5 we immediately obtain

$$\Gamma\varphi = \gamma\nabla\varphi, \qquad \Gamma\varphi_\delta = \gamma\nabla\varphi_\delta \tag{23}$$

$$\Lambda\Phi = \gamma(\nabla \times (\nabla \times \Phi)), \qquad \Lambda\Phi_\delta = \gamma(\neq \times (\nabla \times \Phi_\delta)) \tag{24}$$

In view of Corollary 3 and the relationships (16) and (21)–(24) we have

$$\|\chi_\Omega \Gamma\varphi - \Gamma\varphi_\delta\|_2 \leqslant \|\gamma\, \nabla\varphi - \gamma\, \nabla\varphi_\delta\|_2 + \|\gamma\, \nabla\varphi_\delta\|_2 \leqslant C\gamma_e \delta^{-1}, \qquad \delta > 0$$

$$\|\chi_\Omega \Lambda\Phi - \Lambda\Phi_\delta\|_2 \leqslant \|\gamma(\nabla \times (\nabla \times \Phi)) - \gamma(\nabla \times (\nabla \times \Phi_\delta))\|_2$$
$$+ \|\gamma(\nabla \times (\nabla \times \Phi_\delta))\|_2 \leqslant C\gamma_e \delta^{-1}, \qquad \delta > 0$$

The validity of (20) follows immediately from (16), which completes the proof of the lemma. ■

Proposition 7. Let O be an open bounded domain with regular boundary (see the Definition), and Λ_O and Γ_O be self-adjoint operators defined by (4) and (5), respectively. Let us denote by $D_{\Lambda,O}$ and $D_{\Gamma,O}$ the functions φ and Φ which satisfy the conditions (ii) in Lemmas 4 and 5, respectively. Then the operators Λ_O and Γ_O are essentially self-adjoint on $D_{\Lambda,O}$ and $D_{\Gamma,O}$, respectively.

Proof. If O is smooth, the validity of the proposition follows from ref. 20. If O is a parallelepiped, then all eigenfunctions can be found explicitly and the statement can be justified straightforwardly. ■

Proof of Theorem 4. Suppose first that the conditions of Theorem 1 are satisfied. To prove the strong resolvent convergence of the operators Λ_s and Γ_s we shall apply Lemma 1. We claim that for a sufficiently wide set of functions u and U there exist, respectively, vectors u_s and U_s such that

$$\lim_{s\to\infty} \|u-u_s\|=0, \qquad \lim_{s\to\infty} \|\Gamma^{(0)}u-\Gamma_s u_s\|=0 \tag{25}$$

$$\lim_{s\to\infty} \|U-U_s\|=0. \qquad \lim_{s\to\infty} \|\Lambda^{(0)}U-\Lambda_s U_s\|=0 \tag{26}$$

Notice that in view of (10) and (11) the operators $\Gamma^{(0)}$ and $\Lambda^{(0)}$ are direct sums of the operators Γ_O, Λ_O (where O runs over the set $\{O_\alpha\}$) and the operators $\Gamma_{\mathscr{B}}$, $\Lambda_{\mathscr{B}}$, respectively. Let us fix O and consider $u\in D_{\Gamma,O}$, $U\in D_{\Lambda,O}$ (the sets D are defined in Proposition 7). Being given those u and U, we apply Lemma 6 for $\gamma_i=1$, $\gamma_e=s^{-1}$, $\delta=\delta(s)=s^{-1/2}$, and $u_s=\varphi_{\delta(s)}$, $U_s=\Phi_{\delta(s)}$. Then we just observe that the statements of Lemma 6 imply straightforwardly (25) and (26). If $u\in L^2(\mathscr{B})$, we consider a set of $v_t\in C_0^\infty(\mathscr{B})$ such that $\lim_{t\to\infty}\|u-v_t\|=0$. Then if $a(t)=\sup\{\|\Delta u_\tau\|: 0<\tau\leqslant t\}$, we pick any nondecreasing function $t(s)$ such that $s^{-1}a(t(s))\to 0$ as $s\to\infty$ and set $u_s=v_{t(s)}$. For this choice of u and u_s, (25) is evidently true. The proof of (26) for $U\in L_3^2(\mathscr{B})$ is analogous. Now we define

$$D_\Gamma=\left(\bigcup_\alpha D_{\Gamma,O_\alpha}\right)\bigcup C_0^\infty(\mathscr{B})$$

and D_Λ by the analogous formula and observe the operators $\Gamma^{(0)}$ and $\Lambda^{(0)}$ are essentially self-adjoint on this sets respectively in view of Proposition 7. Then we notice that Lemma 1 is applicable for the operators Λ_s, $\Lambda^{(0)}$ and Γ_s, $\Gamma^{(0)}$, respectively and thus (12) and (13) are true.

If we suppose now that the conditions of Theorem 3 are fulfilled, then the previous arguments hold entirely with the following comments.

1. Being given u or U and employing Lemma 6, we set $\delta=\delta(s)=l(s)/3$ and shift the argument of the relevant functions by $l(s)\,\alpha$, taking in account the simple dependence of O_α on s by the condition (i) of Theorem 3. Then we use (9).

2. We drop the sets $L^2(\mathscr{B})$ and $L_3^2(\mathscr{B})$ from consideration since they degenerate to zero space because of (9).

This completes the proof of Theorem 4. ■

Proof of Proposition 2. Relationship (7) is well known, so we have to establish just the representation (8). If $A = A(x)$, $x \in \mathbf{R}^3$, is a vector function, we denote by $\mathfrak{C}A$ its "curl," i.e., $\mathfrak{C}A = \nabla \times A$. Then we consider the following two eigenvalue problems:

$$N: \begin{cases} \mathfrak{C}^2 \Psi = \lambda \Psi \\ v \times \mathfrak{C}\Psi|_{\partial O} = 0 \end{cases}, \qquad D: \begin{cases} \mathfrak{C}^2 \Psi = \lambda \Psi \\ v \times \Psi|_{\partial O} = 0 \end{cases}$$

where N and D stand, respectively, for the Neumann- and Dirichlet-type problems. In fact, we are interested here mainly in the N-problem. One can easily see that for any differentiable scalar function ψ the vector function $\nabla\psi$ is the solution of the N-problem for $\lambda = 0$. If $\lambda \neq 0$ and Ψ is a smooth enough solution of either the N- or D-problem, then $\nabla \cdot \Psi = 0$. Thus, from now on we shall consider just such Ψ that $\nabla \cdot \Psi = 0$, and $\lambda \neq 0$. Now we notice the following simple connection between the two problems: if Ψ is a smooth enough solution of the *D-problem*, then $\mathfrak{C}\Psi$ is a solution of the *N-problem* and vice versa. Then we recall that O is a cube and consider the following set of vector functions $A(a, \lambda)$,[21]

$$A(a, \mu) = \begin{bmatrix} a_1 \cos \mu_1 x_1 \sin \mu_2 x_2 \sin \mu_3 x_3 \\ a_2 \sin \mu_1 x_1 \cos \mu_2 x_2 \sin \mu_3 x_3 \\ a_3 \sin \mu_1 x_1 \sin \mu_2 x_2 \cos \mu_3 x_3 \end{bmatrix}, \qquad a \cdot \mu = \sum_{j=1}^{3} a_j \mu_j = 0 \tag{27}$$

$$a \in \mathbf{R}^3 \text{ (or } \mathbf{C}^3\text{)}, \qquad \mu/(\pi L) \in \mathbf{Z}^3$$

One can verify straightforwardly that $\mathfrak{C}^2 A(a, \mu) = \mu^2 A(a, \mu)$ and the functions $A(a, \mu)$ satisfy the boundary conditions of the D-problem. On other hand, the closure of the linear span of these functions forms clearly a subspace of $L_3^2(\mathbf{R}^3)$ which is exactly the closure of the linear span of vector functions Ψ such that $\nabla \cdot \Psi = 0$. In view of the connection between the N- and D-problems we may conclude that (8) is true. Besides, the corresponding eigenmodes of the N-problem have the form $\mathfrak{C}A(a, \mu)$ where A, a, and μ satisfy (27) with real a_j and whole $\mu_j/(\pi L)$. This completes the proof of Proposition 2. ■

Proof of Theorems 1 and 3. The statements of these theorems follow from the resolvent convergence[19] of operators Λ and Γ as $s \to \infty$ and Proposition 2.

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